π is an **irrational** number; it has no fraction equivalent and there are no recurring cycles within its infinite string of decimals. Certain **approximations** or **estimates** of π are sufficient for different calculations.

Task one

A commonly used approximation is $\dfrac{22}{7}$, discovered by Archimedes of Syracuse (287 – 212 BC).

  a) Use the division button on your scientific calculator to find its decimal equivalent.

  b) Your calculator may show the answer as a fraction. Which button converts it into a decimal?

You should find the result is $3.\dot{1}4285\dot{7}$. This approximation is only correct for the first two decimal places of π.

  c) Use a scientific calculator to find the decimal equivalent of these approximations of π:

  i. $\dfrac{377}{120}$ *Claudius Ptolemy, circa 85 – 165*

  ii. $\dfrac{864}{275}$ *Leonardo Fibonacci, 1170 – 1250*

  iii. $\dfrac{355}{113}$ *Tsu Ch'ung-Chi, 425 – 501*

  Which approximation gives the best value for π? Explain your answer by looking at the number of correct decimal places.

Task two

Find the decimal equivalent of each of these approximations. If you enter each one into your calculator exactly as shown, you will only need to press the equals button once.

  a) $\left(\dfrac{16}{9}\right)^2$ *Ancient Egypt*

  b) $\dfrac{1}{2\sqrt{2}}\left(\dfrac{99^2}{1103}\right)$ *Srinivasa Ramanujan, 1887 – 1920*

  c) $\left(9^2 + \dfrac{19^2}{22}\right)^{\frac{1}{4}}$ *Ramanujan*

  d) $\dfrac{63}{25}\left(\dfrac{17 + 15\sqrt{5}}{7 + 15\sqrt{5}}\right)$ *Ramanujan*

Which approximation gives the most accurate value of π?

Extension task

This calculation would give us π if continued to infinity. It is called an **infinite series**.

$$4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots\right)$$

Gottfried Wilhelm von Leibniz, 1646 – 1716

a) Explain how the series progresses (i.e. how the calculation continues).

b) Stopping the calculation at different points gives different approximations for π. Find the decimal equivalents of these calculations:

   i. $4\left(1 - \frac{1}{3}\right)$

   ii. $4\left(1 - \frac{1}{3} + \frac{1}{5}\right)$

   iii. $4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}\right)$

   What do you notice about the values as an approximation of π as the series progresses?

c) Your calculator has a limit on calculation length. Find how far it allows you to take this series and comment on this as an approximation of π.

There are many other infinite series which calculate π. Explore these approximations in the same manner:

i. $2\left(\frac{2}{1} \times \frac{2}{3} \times \frac{4}{3} \times \frac{4}{5} \times \frac{6}{5} \times \frac{6}{7} \times \ldots\right)$

John Wallis, 1616 – 1703

ii. $\sqrt{6\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \ldots\right)}$

iii. $4\left(\frac{1}{2 \times 3 \times 4} - \frac{1}{4 \times 5 \times 6} + \frac{1}{6 \times 7 \times 8} - \frac{1}{8 \times 9 \times 10} \ldots\right) + 3$

iv. $\frac{4}{\sqrt{2}}\left(1 + \frac{1}{3^2} - \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} - \frac{1}{13^2} - \frac{1}{15^2} + \ldots\right)$

v. $\sqrt{6\left(\frac{2^2}{2^2 - 1} \times \frac{3^2}{3^2 - 1} \times \frac{5^2}{5^2 - 1} \times \frac{7^2}{7^2 - 1} \times \frac{11^2}{11^2 - 1} \times \ldots\right)}$

Leonhard Euler, 1707 – 1783

A challenge!

$$\frac{2}{\sqrt{\frac{1}{2}} \times \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}} \times \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}} \times \ldots}$$

François Viète, 1540 – 1603

Teaching notes

This resource uses various estimations of π to explore the functions on a scientific calculator. Students are encouraged to us the fraction and index buttons to input the calculations exactly as shown, but you could use the resource to start discussions on:

- breaking a calculation into smaller steps following the correct order of operations. Students could test their answers on basic calculators or use a calculator's memory functions.
- other irrational numbers, such as $\sqrt{2}$, Euler's Number, and the golden ratio
- the existence of infinite series (and convergence)
- the Mathematicians mentioned throughout the resource.

Students will need to compare their answers with π; the first 500 decimal places are given below for photocopying, or values for π are widely available online to project onto an IWB.

Note
This resource was designed using the Casio fx-83 series; you may wish to test the tasks if using other scientific calculators.

500 decimal places of π

3.

1415926535 8979323846 2643383279 5028841971 6939937510

5820974944 5923078164 0628620899 8628034825 3421170679

8214808651 3282306647 0938446095 5058223172 5359408128

4811174502 8410270193 8521105559 6446229489 5493038196

4428810975 6659334461 2847564823 3786783165 2712019091

4564856692 3460348610 4543266482 1339360726 0249141273

7245870066 0631558817 4881520920 9628292540 9171536436

7892590360 0113305305 4882046652 1384146951 9415116094

3305727036 5759591953 0921861173 8193261179 3105118548

0744623799 6274956735 1885752724 8912279381 8301194912